Math 247A Lecture 13 Notes

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1 The Hardy-Littlewood-Sobolev Inequality

1.1 Failure of bounds for L^1 vector-valued maximal function

Let $f : \mathbb{R}^d \to \ell^1$, $f = \{f_n\}_{n \geq 1}$ with $|f(x)| = \sum_{n \geq 1} |f_n(x)|$. We define the L^1 vector-valued maximal function as $\overline{M}_1 f(x) = \sum_{n \geq 1} M f_n(x)$. The the following claims fail:

1. $|\{x : \overline{M}_1 f(x) > \lambda\}| \leq \frac{1}{\lambda} \|f\|_{L^1}$ uniformly for $\lambda > 0, f \in L^1$.
2. For $1 < p < \infty$, $\|\overline{M}_1 f\|_{L^p} \lesssim \|f\|_{L^p}$ uniformly for $f \in L^p$.

Fix $d = 1$. Take $[0, 1]$ and subdibide it into intervals $I_1, \dots, I_N$ of equal length. Let

$$f_n = \begin{cases} \mathbb{1}_{I_n} & 1 \leq n \leq N \\ 0 & n > N. \end{cases}$$

Let $f = \{f_n\}_{n \geq 1}$. Then

$$|f(x)| = \sum_{n \geq 1} |f_n(x)| = \mathbb{1}_{[0,1]}(x) \in L^p \qquad \forall 1 \leq p \leq \infty,$$

$$\|f\|_{L^p} = 1 \qquad \forall 1 \leq p \leq \infty.$$

On the other hand,

$$\begin{aligned} \overline{M}_1 f(x) &= \sum_{n=1}^{N} M f_n(x) \\ &= \sum_{n=1}^{N} \sup_{r > 0} \frac{1}{2r} \int_{x-r}^{x+r} \mathbb{1}_{I_n}(y)\, dy \end{aligned}$$

For $x \in [0, 1]$, $\overline{M} f(x) \geq \sum_{n=1}^{\lfloor N/2 \rfloor} \frac{1}{2(n/N)} \cdot \frac{1}{N} \gtrsim \log(N)$. This tells us that

$$|\{x : \overline{M}_1 f(x) > \frac{1}{10} \log N\}| \geq 1,$$

$$\|\overline{M}_1 f\|_{L^p} \gtrsim \log N.$$

1.2 The Hardy-Littlewood-Sobolev inequality

Theorem 1.1 (Hardy-Littlewood-Sobolev)**.** *Fix $1 < p < r < \infty$ and $1 < q < \infty$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then*

$$\|f * g\|_{L^r} \lesssim \|f\|_p \|g\|^*_{L^{q,\infty}},$$

uniformly for $f \in L^p, g \in L^{q,\infty}$. In particular, for $0 < \alpha < d$,

$$\left\| f * \frac{1}{|x|^\alpha} \right\|_{L^r} \lesssim \|f\|_{L^p},$$

provided $1 + \frac{1}{r} = \frac{1}{p} + \frac{\alpha}{d}$.

Proof. Fix $g \in L^{q,\infty}$. We may assume that $\|g\|^*_{L^{q,\infty}} = 1$. We want to show that the sublinear operator $f \overset{T}{\mapsto} f * g$ is of strong type (p, r). By the Marcinkiewicz interpolation theorem, it suffices to show T is of weak type (p, r) for all $1 < p < r < \infty$ such that $1 + 1/r = 1/p + 1/q$. Say the target is strong-type (p_0, r_0). Then choose $1 < p_1 < p_0 < p_2 < \infty$ and write $\frac{1}{p_0} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. By Marcinkiewicz, if T is of weak-type (p_1, r_1) and (p_2, r_2), then T is of strong-type $(p_0, \widetilde{r})$, where

$$\frac{1}{\widetilde{r}} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2} = \theta \left[\frac{1}{p_1} + \frac{1}{q} + -1\right] + (1-\theta)\left[\frac{1}{p_2} + \frac{1}{q} - 1\right] = \frac{1}{p_0} + \frac{1}{q} + 1 = \frac{1}{r_0}.$$

Let's show T is of weak-type (p, r):

$$|\{x : |(f * g)(x)| > \lambda\}| \lesssim \left(\frac{\|f\|_p}{\lambda}\right)^r$$

We may rescale so that $\|f\|_p = 1$. Write $g = g_1 + g_2 = g\mathbb{1}_{\{|g| \le R\}} + g\mathbb{1}_{\{|g| > R\}}$. Then

$$|\{x : |(f * g)(x)| > \lambda\}| \lesssim |\{x : |(f * g_1)(x)| > \lambda/2\}| + |\{x : |(f * g_2)(x)| > \lambda/2\}|$$

By Chebyshev,

$$\begin{aligned}
|\{x : |(f * g_1)(x)| > \lambda/2\}| &\lesssim \frac{\|f * g_1\|_s^s}{\lambda^s} \\
&\lesssim \frac{\|f\|_p^s \|g_1\|^s_{ps/(ps+p-s)}}{\lambda^s} \\
&\lesssim \lambda^{-s} \left(\int_0^\infty \alpha^{ps/(ps+p-s)} \mathbb{1}_{\{|g_1| > \alpha\}} \frac{d\alpha}{\alpha}\right)^{(ps+p-s)/p} \\
&= \lambda^{-s} \left(\int_0^R \alpha^{ps/(ps+p-s)} \mathbb{1}_{\{|g| > \alpha\}} \frac{d\alpha}{\alpha}\right)^{(ps+p-s)/p} \\
&= \lambda^{-s} (\|g\|^*_{L^{q,\infty}})^{q \cdot (ps+p-s)/p} \left(\int_0^R \alpha^{ps/(ps+p-s) - q} \frac{d\alpha}{\alpha}\right)^{(ps+p-s)/p}
\end{aligned}$$

$$\lesssim \lambda^{-s} R^{s-q\cdot(ps+p-s)/p},$$

Provided $\frac{1}{q} > 1 + \frac{1}{s} - \frac{1}{p}$. Since $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}$, this means that $s > r$.

On the other hand, by Chebyshev,

$$\begin{aligned} |\{x : |(f * g_2)(x)| > \lambda/2\}| &\lesssim \frac{\|f * g_2\|_p^p}{\lambda^p} \\ &\lesssim \frac{\|f\|_p^p \|g_2\|_1^p}{\lambda^p} \\ &\lesssim \lambda^{-p} \left(\int_0^\infty |\{x : |g_2|(x) > \alpha\}| \, d\alpha \right)^p \\ &\lesssim \lambda^{-p} \left(\|g\|_{L^{q,\infty}}^*\right)^{pq} \left(\int_R^\infty \alpha^{-q} \, dx \right)^p \\ &\lesssim \lambda^{-p} R^{(1-q)p}. \end{aligned}$$

Optimize in R:

$$\lambda^{-s} R^{s-qs(1+1/s-1/p)} = \lambda^{-p} R^{(1-q)p}$$
$$\lambda^{p-s} = R^{(1-q)(p-s)} R^{q/p\cdot(p-s)}$$
$$\lambda = R^{1-q+q/p} = R^{q(1/q+1/p-1)} = R^{q/r}.$$

So we optimize at $R = \lambda^{r/q}$.

So

$$\begin{aligned} |\{x : |f * g|(x) > \lambda\}| &\lesssim \lambda^{-p} \lambda^{r/q\cdot p(1-q)} \\ &\lesssim \lambda^{-p(1-r/q+r)} \\ &\lesssim \lambda^{-pr(1/r-1/q+1)} \\ &\lesssim \lambda^{-pr/p} \\ &\lesssim \lambda^{-r}. \end{aligned}$$

□

Although we have just proven this claim, here is Hedberg's proof of $\|f * \frac{1}{|x|^\alpha}\|_r \lesssim \|f\|_p$ whenever $0 < \alpha < d$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{\alpha}{d}$.

Proof. Fix $x \in \mathbb{R}^d$. Then

$$\left(f * \frac{1}{|x|^\alpha} \right)(x) = \int \frac{f(y)}{|x-y|^\alpha} \, dy = \int_{|x-y|\le R} \frac{f(y)}{|x-y|^\alpha} \, dy + \int_{|x-y|>R} \frac{f(y)}{|x-y|^\alpha} \, dy.$$

$$
\begin{aligned}
\left|\int_{|x-y|\le R} \frac{f(y)}{|x-y|^\alpha}\,dy\right| &\le \sum_{\substack{r\in 2^{\mathbb{Z}}\\ r\le R}} \int_{R\le|x-y|\le 2r} \frac{f(y)}{|x-y|^\alpha}\,dy \\
&\lesssim \sum_{r\le R} r^{-\alpha} r^d \frac{1}{|B(x,2r)|} int_{B(x,2r)} |f(y)|\,dy \\
&\lesssim R^{d-\alpha} Mf(x).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\left|\int_{|x-y|>R} \frac{f(y)}{|x-y|^\alpha}\,dy\right| &= \left| f * \frac{\mathbb{1}_{\{|x|>R\}}}{|x|^\alpha}\right|(x) \\
&\lesssim \|f\|_p \left\| \frac{\mathbb{1}_{\{|x|>R\}}}{|x|^\alpha}\right\|_{p'} \\
&\lesssim \|f\|_p \int_R^\infty \frac{r^{d-1}}{r^{\alpha p'}}\,dr \\
&\lesssim \|f\|_p R^{d/p'-\alpha} \\
&\lesssim \|f\|_p R^{d(1-1/p-\alpha/d)} \\
&\lesssim \|f\|_p R^{-d/r}.
\end{aligned}
$$

Optimize in R: choose

$$
R^{d-\alpha} Mf(x) = \|f\|_p R^{-d/r}
$$

$$
R^{d/p} = R^{d(1-\alpha/d+1/r)} = \frac{\|f\|_p}{Mf(x)}.
$$

So

$$
\left|\left(f * \frac{1}{|x|^\alpha}\right)(x)\right| \lesssim \|f\|_p \left(\frac{\|f\|_p}{Mf(x)}\right)^{-p/r} \lesssim Mf(x)^{p/r} \|f\|_p^{1-p/r}.
$$

So

$$
\begin{aligned}
\left\| f * \frac{1}{|x|^\alpha}\right\|_r &\lesssim \|f\|_p^{1-p/r} \|(Mf)^{p/r}\|_r \\
&\lesssim \|f\|_p^{1-p/r} \|Mf\|_p^{p/r} \\
&\lesssim \|f\|_p. \qquad \square
\end{aligned}
$$